Derivace funkcí více proměnných

Definice. *Parciální derivace funkce $f(\boldsymbol{x})$ podle x_i* v bodě $\boldsymbol{a} \in D(f)$:

$$\frac{\partial f}{\partial x_i}(\boldsymbol{a}) = \varphi'(a_i)$$

pro $\varphi(x_i) = f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$.

Příklad. $f(x,y) = \mathrm{e}^x + x^2 y$, $\boldsymbol{a} = (1,2)$
$\frac{\partial f}{\partial x}(x,y) = \mathrm{e}^x + 2xy$, $\frac{\partial f}{\partial x}(1,2) = \mathrm{e} + 4$
$\frac{\partial f}{\partial y}(x,y) = x^2$, $\frac{\partial f}{\partial y}(1,2) = 1$

Definice. *Gradient* funkce f v bodě $\boldsymbol{a} \in D(f)$:

$$\operatorname{grad} f(\boldsymbol{a}) = \left(\tfrac{\partial f}{\partial x_1}(\boldsymbol{a}), \dots, \tfrac{\partial f}{\partial x_n}(\boldsymbol{a})\right).$$

Příklad. $f(x,y) = \mathrm{e}^x + x^2 y$
$\operatorname{grad} f(x,y) = (\mathrm{e}^x + 2xy, x^2)$, $\operatorname{grad} f(1,2) = (\mathrm{e}+4, 1)$

Poznámka. $(\operatorname{grad} f)$: $\boldsymbol{a} \in \mathbb{R}^n \mapsto \operatorname{grad} f(\boldsymbol{a}) \in \mathbb{R}^n$ je vektorová funkce

$$(\operatorname{grad} f)(\boldsymbol{a}) = \left(\tfrac{\partial f}{\partial x_1}, \dots, \tfrac{\partial f}{\partial x_n}\right)(\boldsymbol{a})$$
$$\operatorname{grad} f = \left(\tfrac{\partial f}{\partial x_1}, \dots, \tfrac{\partial f}{\partial x_n}\right) = \left(\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}\right) f$$
$$\operatorname{grad} = \left(\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}\right) = \nabla \quad (\textit{nabla})$$

$$\frac{\partial f}{\partial x_i}(\boldsymbol{a}) = \lim_{t\to 0} \frac{\varphi(a_i + t) - \varphi(a_i)}{t} = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}_i) - f(\boldsymbol{a})}{t}$$

Definice. *Derivace funkce f ve směru $\boldsymbol{h} \in \mathbb{R}^n$* v bodě $\boldsymbol{a} \in D(f)$:

$$f'_{\boldsymbol{h}}(\boldsymbol{a}) = \lim_{t\to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{h}) - f(\boldsymbol{a})}{t}.$$

Příklad. $f(x,y) = \mathrm{e}^x + x^2 y$, $\boldsymbol{a} = (1,2)$, $\boldsymbol{h} = (-1,3)$
$f'_{(-1,3)}(1,2) = \lim_{t\to 0}\left(\frac{1}{t}(\mathrm{e}^{1-t} - 1) - 1 - 4t + 3t^2\right) = -\mathrm{e} - 1$

Poznámky.
1) Parciální derivace je směrová: $\frac{\partial f}{\partial x_i} = f'_{\boldsymbol{e}_i}$.
2) Někdy se uvažují jen jednotkové vektory $\boldsymbol{h}$: $\|\boldsymbol{h}\| = 1$.
3) $f'_{\boldsymbol{o}}(\boldsymbol{a}) = 0$.
4) $f'_{c\boldsymbol{h}}(\boldsymbol{a}) = c \cdot f'_{\boldsymbol{h}}(\boldsymbol{a})$.
5) $\varphi(t) = f(\boldsymbol{a} + t\boldsymbol{h})$: $f'_{\boldsymbol{h}}(\boldsymbol{a}) = \lim_{t\to 0} \frac{\varphi(t) - \varphi(0)}{t} = \varphi'(0)$.

Věta. *Nechť $\boldsymbol{a}, \boldsymbol{h} \in \mathbb{R}^n$ a existují $f'_{\boldsymbol{h}}(\boldsymbol{a}), g'_{\boldsymbol{h}}(\boldsymbol{a})$. Pak*

1) $(f+g)'_{\boldsymbol{h}}(\boldsymbol{a}) = f'_{\boldsymbol{h}}(\boldsymbol{a}) + g'_{\boldsymbol{h}}(\boldsymbol{a})$,
2) $(f \cdot g)'_{\boldsymbol{h}}(\boldsymbol{a}) = f'_{\boldsymbol{h}}(\boldsymbol{a}) \cdot g(\boldsymbol{a}) + f(\boldsymbol{a}) \cdot g'_{\boldsymbol{h}}(\boldsymbol{a})$,
3) $\left(\frac{f}{g}\right)'_{\boldsymbol{h}}(\boldsymbol{a}) = \dfrac{f'_{\boldsymbol{h}}(\boldsymbol{a}) \cdot g(\boldsymbol{a}) - f(\boldsymbol{a}) \cdot g'_{\boldsymbol{h}}(\boldsymbol{a})}{g(\boldsymbol{a})^2}$ *$(g(\boldsymbol{a}) \neq 0)$.*

Důkaz. Přepisem do jedné proměnné.

Věta. *Nechť $\boldsymbol{a}$ je vnitřní bod $D(f)$, parciální derivace f existují v některém okolí $\boldsymbol{a}$ a jsou v tomto bodě spojité (tj. $\operatorname{grad} f$ je spojitý v $\boldsymbol{a}$). Pak*

$$f'_h(\boldsymbol{a}) = \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}.$$

Příklad. $f(x,y) = \mathrm{e}^x + x^2 y$, $\operatorname{grad} f(1,2) = (\mathrm{e}+4, 1)$
$f'_{(-1,3)}(1,2) = (\mathrm{e}+4, 1) \cdot (-1,3) = -\mathrm{e} - 1$

Příklad. Předpoklad o spojitosti nelze vypustit.
$f(x,y) = \frac{xy}{x^2+y^2}$ pro $(x,y) \neq (0,0)$, $f(0,0) = 0$
$\operatorname{grad} f(0,0) = (0,0)$ $(f(x,0) = f(0,y) = 0)$
$f'_{(1,1)}(0,0) = \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{0{,}5 - 0}{t}$ neexistuje
$\frac{\partial f}{\partial x}(x,y) = \frac{y(y^2 - x^2)}{(x^2+y^2)^2}$
$\lim_{\substack{(x,y)\to(0,0)\\ y = 2x}} \frac{\partial f}{\partial x}(x,y) = \lim_{x\to 0} \frac{6}{25x}$ neexistuje

Věta (Lagrange). *Nechť $I \subset D(f)$ je úsečka s krajními body $\boldsymbol{a}$ a $\boldsymbol{b}$, f je spojitá na I a má v každém bodě $I \setminus \{\boldsymbol{a}, \boldsymbol{b}\}$ derivaci ve směru $\boldsymbol{b} - \boldsymbol{a}$. Pak existuje $\alpha \in (0,1)$ tak, že*

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = f'_{\boldsymbol{b}-\boldsymbol{a}}\big(\boldsymbol{a} + \alpha(\boldsymbol{b} - \boldsymbol{a})\big).$$

Důkaz. $f\big(\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})\big) = \varphi(t)$, $t \in \langle 0, 1\rangle$
$f(\boldsymbol{a}) = \varphi(0)$, $f(\boldsymbol{b}) = \varphi(1)$, $f'_{\boldsymbol{b}-\boldsymbol{a}}\big(\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})\big) = \varphi'(t)$
Lagrange $n = 1$: $\varphi(1) - \varphi(0) = \varphi'(\alpha)\,(1-0) = \varphi'(\alpha)$

Poznámka. Bod $\boldsymbol{a} + \alpha(\boldsymbol{b} - \boldsymbol{a})$ leží uvnitř úsečky I.

Příklad. $f(x,y) = 1$ na $\{(x,y) \colon y = x^2,\ x \neq 0\}$, jinak 0
$f'_{\boldsymbol{h}}(0,0) = 0$, f není spojitá v $(0,0)$

lineární aproximace přírůstku: $f(a+h) - f(a) = k \cdot h + \omega(h)$
$\frac{\omega(h)}{h} = \frac{f(a+h) - f(a) - kh}{h}$
$\lim_{h\to 0} \frac{\omega(h)}{h} = f'(a) - k$ ... nejlepší pro $k = f'(a)$
nejlepší lineární aproximace $L(h) = f'(a)\,h$
$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0$

Definice. (*Totální*) *diferenciál* funkce f v bodě $\boldsymbol{a}$ (vnitřní bod $D(f)$) je lineární zobrazení $L \colon \mathbb{R}^n \to \mathbb{R}$, pro které platí

$$\lim_{\boldsymbol{h}\to\boldsymbol{o}} \frac{f(\boldsymbol{a} + \boldsymbol{h}) - f(\boldsymbol{a}) - L(\boldsymbol{h})}{\|\boldsymbol{h}\|} = 0.$$

Pokud existuje, říkáme, že f je *diferencovatelná* v bodě $\boldsymbol{a}$.

Poznámky. 1) Značení: $\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h})$, $\mathrm{d}f(\boldsymbol{a})[\boldsymbol{h}]$, $\mathrm{d}f(\boldsymbol{a}, \boldsymbol{h})$.
2) $\mathrm{d}f(\boldsymbol{a})$ je lineární zobrazení:

$$\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h}) = k_1 h_1 + \dots + k_n h_n$$
$$\mathrm{d}f(\boldsymbol{a}) = k_1\,\mathrm{d}x_1 + \dots + k_n\,\mathrm{d}x_n = (k_1, \dots, k_n) \cdot \mathrm{d}\boldsymbol{x}$$

Příklad. $f(x,y) = x^2 + y^2$, $\mathrm{d}f(1,1)(h_1, h_2) = 2h_1 + 2h_2$.

Poznámky. 1) Je-li f lineární, pak $\mathrm{d}f(\boldsymbol{a}) = f$.
2) $n = 1$: $\mathrm{d}f(a)(h) = f'(a)\,h$, $\mathrm{d}f(a)$ s $f'(a)$ ztotožňujeme.
3) Pro vektorovou funkci $F = (f_1, \dots, f_n) \colon \mathbb{R}^n \to \mathbb{R}^k$ je $\mathrm{d}F = (\mathrm{d}f_1, \dots, \mathrm{d}f_n)$, tj.:

$$\lim_{\boldsymbol{h}\to\boldsymbol{o}} \frac{\|F(\boldsymbol{a} + \boldsymbol{h}) - F(\boldsymbol{a}) - \mathrm{d}F(\boldsymbol{a})(\boldsymbol{h})\|}{\|\boldsymbol{h}\|} = 0.$$

Věty o derivacích

Věta. *Má-li funkce f v bodě $\boldsymbol{a}$ totální diferenciál, pak má v $\boldsymbol{a}$ všechny směrové derivace a platí (pro každé $\boldsymbol{h} \in \mathbb{R}^n$)*

$$f'_{\boldsymbol{h}}(\boldsymbol{a}) = \mathrm{d}f(\boldsymbol{a})(\boldsymbol{h}) = \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}\,.$$

Důkaz. 1) $\boldsymbol{h} = \boldsymbol{o}$: $f'_{\boldsymbol{o}}(\boldsymbol{a}) = 0 = \mathrm{d}f(\boldsymbol{a})(\boldsymbol{o})$
$\boldsymbol{h} \neq \boldsymbol{o}$: $t \to 0 \iff t\boldsymbol{h} \to \boldsymbol{o}$:

$$f'_{\boldsymbol{h}}(\boldsymbol{a}) - \mathrm{d}f(\boldsymbol{a})(\boldsymbol{h}) = \lim_{t\to 0} \frac{f(\boldsymbol{a}+t\boldsymbol{h}) - f(\boldsymbol{a}) - t\,\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h})}{t}$$
$$= \lim_{t\boldsymbol{h}\to\boldsymbol{o}} \underbrace{\frac{f(\boldsymbol{a}+t\boldsymbol{h}) - f(\boldsymbol{a}) - \mathrm{d}f(\boldsymbol{a})(t\boldsymbol{h})}{\pm\|t\boldsymbol{h}\|}}_{\to 0} \cdot \|\boldsymbol{h}\| = 0$$

2) $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ standardní ortonormální báze $\mathbb{R}^n$:

$$\begin{aligned}
\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h}) &= \mathrm{d}f(\boldsymbol{a})(h_1\boldsymbol{e}_1 + \dots + h_n\boldsymbol{e}_n)\\
&= h_1\,\mathrm{d}f(\boldsymbol{a})(\boldsymbol{e}_1) + \dots + h_n\,\mathrm{d}f(\boldsymbol{a})(\boldsymbol{e}_n)\\
&\overset{1)}{=} h_1\, f'_{\boldsymbol{e}_1}(\boldsymbol{a}) + \dots + h_n\, f'_{\boldsymbol{e}_n}(\boldsymbol{a})\\
&= h_1 \tfrac{\partial f}{\partial x_1}(\boldsymbol{a}) + \dots + h_n \tfrac{\partial f}{\partial x_n}(\boldsymbol{a})\\
&= \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}
\end{aligned}$$

Poznámka. Stručné zápisy:

$$\begin{aligned}
\mathrm{d}f(\boldsymbol{a}) &= \operatorname{grad} f(\boldsymbol{a}) \cdot \mathrm{d}\boldsymbol{x}\\
\mathrm{d}f &= \mathrm{d}\boldsymbol{x} \cdot \operatorname{grad} f\\
\mathrm{d} &= \mathrm{d}\boldsymbol{x} \cdot \operatorname{grad}
\end{aligned}$$

Poznámka. Je-li $A\colon \mathbb{R}^n \to \mathbb{R}^k$ lineární, pak existuje matice $\boldsymbol{A}$ typu (k,n) tak, že

$$A(\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{A} \cdot \boldsymbol{x}^{\mathrm{T}}\,.$$

Diferenciál $\mathrm{d}f(\boldsymbol{a})$ má za matici $\operatorname{grad} f(\boldsymbol{a})$.

$$\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h})^{(\mathrm{T})} = \left(\tfrac{\partial f}{\partial x_1}(\boldsymbol{a}) \ \dots\ \tfrac{\partial f}{\partial x_n}(\boldsymbol{a})\right) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.$$

Pro vektorovou funkci $F\colon \mathbb{R}^n \to \mathbb{R}^k$, $F = (f_1, \dots, f_k)$ máme diferenciály po souřadnicích:

$$\mathrm{d}F(\boldsymbol{a})(\boldsymbol{h}) = \big(\mathrm{d}f_1(\boldsymbol{a})(\boldsymbol{h}), \cdots, \mathrm{d}f_k(\boldsymbol{a})(\boldsymbol{h})\big)$$
$$\mathrm{d}F(\boldsymbol{a})(\boldsymbol{h})^{\mathrm{T}} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{a}) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_k}{\partial x_n}(\boldsymbol{a}) \end{pmatrix}}_{\textit{Jacobiho matice } F \text{ v } \boldsymbol{a}} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

Příklad. $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (x^2 + xy, 2x + 5y)$

$$\mathrm{d}F(x,y) \sim \begin{pmatrix} 2x+y & x \\ 2 & 5 \end{pmatrix}$$
$$\mathrm{d}F(1,1)(h_1,h_2)^{\mathrm{T}} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 3h_1 + h_2 \\ 2h_1 + 5h_2 \end{pmatrix}$$
$$\mathrm{d}F(1,1)(h_1,h_2) = (3h_1 + h_2, 2h_1 + 5h_2)$$

Věta. *Má-li funkce v některém bodě diferenciál, pak je v tomto bodě spojitá.*

Důkaz. $\|\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h})\| = \|\operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}\| \leq \|\operatorname{grad} f(\boldsymbol{a})\| \cdot \|\boldsymbol{h}\|$:
$\lim_{\boldsymbol{h}\to\boldsymbol{o}}[f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a})] =$
$= \lim_{\boldsymbol{h}\to\boldsymbol{o}}\Big[\underbrace{\tfrac{f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - \mathrm{d}f(\boldsymbol{a})(\boldsymbol{h})}{\|\boldsymbol{h}\|}}_{\to 0} \cdot \|\boldsymbol{h}\| + \underbrace{\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h})}_{\to 0}\Big] = 0.$

Věta. *Má-li funkce v některém vnitřním bodě svého definičního oboru spojité parciální derivace (tj. spojitý gradient), pak má v tomto bodě diferenciál.*

Důkaz. Ověříme, že $\operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}$ je diferenciál:

$$\begin{aligned}
&f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h} =\\
&= [f(a_1+h_1, a_2+h_2, \dots) - f(a_1, a_2+h_2, \dots, a_n+h_n) +\\
&+ [f(a_1, a_2+h_2, \dots, a_n+h_n) - f(a_1, a_2, \dots, a_n+h_n) + \cdots\\
&+ [f(a_1, a_2, \dots, a_n+h_n) - f(a_1, a_2, \dots, a_n) - \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}\\
&= \tfrac{\partial f}{\partial x_1}\big(\boldsymbol{x}_1(\boldsymbol{h})\big)\, h_1 + \dots + \tfrac{\partial f}{\partial x_n}\big(\boldsymbol{x}_n(\boldsymbol{h})\big)\, h_n -\\
&- \tfrac{\partial f}{\partial x_1}(\boldsymbol{a})\, h_1 - \dots - \tfrac{\partial f}{\partial x_n}(\boldsymbol{a})\, h_n\\
&= \|\boldsymbol{h}\| \sum_{i=1}^n \underbrace{\Big[\tfrac{\partial f}{\partial x_i}\big(\boldsymbol{x}_i(\boldsymbol{h})\big) - \tfrac{\partial f}{\partial x_i}(\boldsymbol{a})\Big]}_{\to 0} \cdot \underbrace{\tfrac{h_i}{\|\boldsymbol{h}\|}}_{\in\langle -1,1\rangle}
\end{aligned}$$

Důsledek. *Jsou-li parciální derivace spojité na otevřené množině, pak diferenciál existuje v každém bodě této množiny.*

Poznámky.
1) Parciální derivace v okolí, spojité v bodě $\Rightarrow$ diferenciál v bodě $\Rightarrow$ všechny směrové derivace v bodě.
2) Podmínka spojitosti parciálních derivací není nutná.

Příklad. $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2}$ pro $(x,y) \neq (0,0)$, $f(0,0) = 0$
diferenciál $\mathrm{d}f(0,0)(h_1,h_2) = 0$ existuje
limita $\frac{\partial f}{\partial x}(x,y) = 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}$ neexistuje

Věta. *Funkce, která má v oblasti $G \subset \mathbb{R}^n$ nulové všechny parciální derivace, je v této oblasti konstantní.*

Důkaz. Zvolme $\boldsymbol{x}, \boldsymbol{y} \in G$, existuje čára $L \subset G$ z $\boldsymbol{x}$ do $\boldsymbol{y}$ složená z úseček, BÚNO z jedné, existuje $\boldsymbol{z} \in L$:
$f(\boldsymbol{y}) - f(\boldsymbol{x}) = f'_{\boldsymbol{y}-\boldsymbol{x}}(\boldsymbol{z}) = \operatorname{grad} f(\boldsymbol{z}) \cdot (\boldsymbol{y}-\boldsymbol{x}) = \boldsymbol{o} \cdot (\boldsymbol{y}-\boldsymbol{x}) = 0.$

Důsledek. *Funkce se stejnými parciálními derivacemi v oblasti se v této oblasti liší o konstantu.*

Příklad. $f(x,y) = \operatorname{arctg} x + \operatorname{arctg} y$

$g(x,y) = \operatorname{arctg}\frac{x+y}{1-xy}$, $xy \neq 1$

$\operatorname{grad} f(x,y) = \left(\frac{1}{1+x^2}, \frac{1}{1+y^2}\right) = \operatorname{grad} g(x,y)$

$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg}\frac{x+y}{1-xy}$ pro $xy < 1$

$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg}\frac{x+y}{1-xy} + \pi$ pro $xy > 1$, $x > 0$

$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg}\frac{x+y}{1-xy} - \pi$ pro $xy > 1$, $x < 0$

Interpretace a aplikace

Směr největšího spádu

pro $\|\boldsymbol{h}\| = 1$ je (Schwarzova nerovnost):
$|f'_{\boldsymbol{h}}(\boldsymbol{a})| = |\operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h}| \leq \|\operatorname{grad} f(\boldsymbol{a})\| \cdot \|\boldsymbol{h}\| = \|\operatorname{grad} f(\boldsymbol{a})\|$
rovnost nastane, pokud jsou $\boldsymbol{h}$ a $\operatorname{grad} f(\boldsymbol{a})$ lineárně závislé, tj. v případě $\operatorname{grad} f(\boldsymbol{a}) \neq 0$ pro jednotkové vektory:

$$\boldsymbol{h}_{\max} = \frac{\operatorname{grad} f(\boldsymbol{a})}{\|\operatorname{grad} f(\boldsymbol{a})\|}, \qquad \boldsymbol{h}_{\min} = -\boldsymbol{h}_{\max}.$$

Tečná nadrovina a normála grafu

tečná nadrovina (lineární aproximace) v $[\boldsymbol{a}, f(\boldsymbol{a})]$:
$y = f(\boldsymbol{a}) + \operatorname{grad} f(\boldsymbol{a}) \cdot (\boldsymbol{x} - \boldsymbol{a})$
$\operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{x} - y = \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{a} - f(\boldsymbol{a})$
$\big(\operatorname{grad} f(\boldsymbol{a}), -1\big) \cdot (\boldsymbol{x}, y) = \big(\operatorname{grad} f(\boldsymbol{a}), -1\big) \cdot \big(\boldsymbol{a}, f(\boldsymbol{a})\big)$
$\big(\operatorname{grad} f(\boldsymbol{a}), -1\big)$ je normálový vektor

Lineární aproximace

$f(\boldsymbol{x}) \approx f(\boldsymbol{a}) + \mathrm{d}f(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}) = f(\boldsymbol{a}) + \operatorname{grad} f(\boldsymbol{a}) \cdot (\boldsymbol{x} - \boldsymbol{a})$

Příklad. Aproximujte $f(x, y) = \operatorname{arctg} \frac{x+y}{1-xy}$ v okolí $(0, 0)$.
$\operatorname{grad} f(x, y) = \big(\frac{1}{1+x^2}, \frac{1}{1+y^2}\big)$, $\operatorname{grad} f(0, 0) = (1, 1)$
$f(x, y) \approx f(0, 0) + (1, 1) \cdot (x, y) = x + y$

Derivace vyšších řádů

Parciální derivace vyšších řádů:

$$\boldsymbol{x} \mapsto \frac{\partial f}{\partial x_i}(\boldsymbol{x})$$
$$\boldsymbol{x} \mapsto \frac{\partial \frac{\partial f}{\partial x_i}}{\partial x_j}(\boldsymbol{x}) = \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}(\boldsymbol{x}) = \frac{\partial^2 f}{\partial x_j \, \partial x_i}(\boldsymbol{x})$$

$\frac{\partial^3 f}{\partial x \, \partial x \, \partial y}(\boldsymbol{x}) = \frac{\partial^3 f}{\partial x^2 \, \partial y}(\boldsymbol{x})$
$\frac{\partial^k f}{\partial x_{i_k} \ldots \partial x_{i_1}}(\boldsymbol{x}) \ldots$ *řádu* k
smíšená: alespoň 2 proměnné různé

Poznámka. Pořadí derivování nelze vždy zaměnit.

Příklad.

$$\begin{aligned}
f(x, y) &= \tfrac{xy(x^2-y^2)}{x^2+y^2} \quad \text{pro } (x, y) \neq (0, 0),\ f(0, 0) = 0 \\
\tfrac{\partial f}{\partial x}(x, y) &= \tfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} \quad \text{pro } (x, y) \neq (0, 0) \\
\tfrac{\partial f}{\partial x}(0, 0) &= 0 \\
\tfrac{\partial f}{\partial y}(x, y) &= \tfrac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} \quad \text{pro } (x, y) \neq (0, 0) \\
\tfrac{\partial f}{\partial y}(0, 0) &= 0 \\
\tfrac{\partial^2 f}{\partial y \, \partial x}(0, 0) &= \lim_{x \to 0} \tfrac{1}{x} \big(\tfrac{\partial f}{\partial x}(0, y) - \tfrac{\partial f}{\partial x}(0, 0)\big) = -1 \\
\tfrac{\partial^2 f}{\partial x \, \partial y}(0, 0) &= \lim_{y \to 0} \tfrac{1}{y} \big(\tfrac{\partial f}{\partial y}(x, 0) - \tfrac{\partial f}{\partial y}(0, 0)\big) = 1
\end{aligned}$$

Věta. *Je-li* $\frac{\partial^2 f}{\partial x_j \, \partial x_i}$ *spojitá na otevřené množině* G*, pak na* G *platí*

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} = \frac{\partial^2 f}{\partial x_j \, \partial x_i}.$$

Věta. *Existují-li* $\frac{\partial^2 f}{\partial x_j \, \partial x_i}$, $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$ *v okolí bodu* $\boldsymbol{a}$ *a jsou-li spojité v* $\boldsymbol{a}$*, pak jsou v tomto bodě stejné.*

Příklad. $f(x, y) = \mathrm{e}^{xy^2}$
$\frac{\partial f}{\partial x}(x, y) = y^2 \mathrm{e}^{xy^2}$, $\frac{\partial f}{\partial y}(x, y) = 2x\mathrm{e}^{xy^2}$
$\frac{\partial^2 f}{\partial y \, \partial x}(x, y) = 2y\mathrm{e}^{xy^2} + 2xy^3 \mathrm{e}^{xy^2} = \frac{\partial^2 f}{\partial x \, \partial y}(x, y)$

Definice. Nechť $G \subset \mathbb{R}^n$ je otevřená, $k \in \mathbb{N}$. Funkce $f \colon G \to \mathbb{R}^n$ se nazývá *třídy* C^k na G ($f \in C^k(G)$), jestliže všechny parciální derivace řádu k jsou na G spojité.

Poznámky.
1) $C^0 \ldots$ spojité funkce
2) $C^0 \supset C^1 \supset C^2 \supset \cdots$
3) $\bigcap_{k=1}^{\infty} C^k = C^{\infty}$

Důsledek. *Je-li* $f \in C^k(G)$*, pak parciální derivace do řádu* k *na* G *nezávisejí na pořadí derivování.*

$$\begin{aligned}
&\mathrm{d}f(\boldsymbol{a})(\boldsymbol{h}) = \operatorname{grad} f(\boldsymbol{a}) \cdot \boldsymbol{h} = (\boldsymbol{h} \cdot \operatorname{grad}) f(\boldsymbol{a}) \\
&\quad \mathrm{d} \to \boldsymbol{h} \cdot \operatorname{grad} = \textstyle\sum_{i=1}^n h_i \frac{\partial}{\partial x_i} \\
&\quad \mathrm{d}^2 \to (\boldsymbol{h} \cdot \operatorname{grad})^2 = \textstyle\sum_{i,j=1}^n h_i h_j \frac{\partial^2}{\partial x_i \, \partial x_j} \\
&\quad \mathrm{d}^3 \to (\boldsymbol{h} \cdot \operatorname{grad})^3 = \textstyle\sum_{i,j,k=1}^n h_i h_j h_k \frac{\partial^3}{\partial x_i \, \partial x_j \, \partial x_k} \\
&\quad \ldots
\end{aligned}$$

Poznámky.
1) Pro $n = 1$ je $\mathrm{d}^k f(a)(h) = f^{(k)}(a) \, h^k$.
2) Pro $f(x, y)$ se spojitými parciálními derivacemi:

$$\begin{aligned}
&\mathrm{d}^2 \to h_1^2 \tfrac{\partial^2}{\partial x^2} + 2h_1h_2 \tfrac{\partial^2}{\partial x \, \partial y} + h_2^2 \tfrac{\partial^2}{\partial y^2} \\
&\mathrm{d}^3 \to h_1^3 \tfrac{\partial^3}{\partial x^3} + 3h_1^2h_2 \tfrac{\partial^3}{\partial x^2 \, \partial y} + 3h_1h_2^2 \tfrac{\partial^3}{\partial x \, \partial y^2} + h_2^3 \tfrac{\partial^3}{\partial y^3} \\
&\ldots
\end{aligned}$$

3)

$$\mathrm{d}^2 \to (h_1, \ldots, h_n) \cdot \underbrace{\begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \cdots & \frac{\partial^2}{\partial x_1 \, \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2}{\partial x_n \, \partial x_1} & \cdots & \frac{\partial^2}{\partial x_n^2} \end{pmatrix}}_{\textit{Hessova matice}} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

Příklad. Pro $f(x, y) = x^y = \mathrm{e}^{y \ln x}$, $D(f) = (0, \infty) \times \mathbb{R}$ je $\mathrm{d}^2 f(1, 2)(h_1, h_2) = 2h_1^2 + 2h_1h_2$.

Taylorův polynom

$$f(a+h) = f(a) + f'(a)\,h + \frac{f''(a)}{2!}\,h^2 + \cdots + \frac{f^{(k)}(a)}{k!}\,h^k + \\ + \frac{f^{(k+1)}(a+\alpha h)}{(k+1)!}\,h^{k+1}$$

Věta (o Taylorově polynomu). *Je-li funkce f třídy C^{k+1} na otevřené množině $G \subset \mathbb{R}^n$ obsahující úsečku s krajními body $\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{h}$, pak existuje $\alpha \in (0,1)$ tak, že platí*

$$f(\boldsymbol{a}+\boldsymbol{h}) = f(\boldsymbol{a}) + \mathrm{d}f(\boldsymbol{a})(\boldsymbol{h}) + \frac{\mathrm{d}^2 f(\boldsymbol{a})(\boldsymbol{h})}{2!} + \cdots + \\ + \frac{\mathrm{d}^{(k)} f(\boldsymbol{a})(\boldsymbol{h})}{k!} + \frac{\mathrm{d}^{(k+1)} f(\boldsymbol{a}+\alpha\boldsymbol{h})(\boldsymbol{h})}{(k+1)!}\,.$$

Poznámky.
1) $\boldsymbol{h} \to \boldsymbol{x} - \boldsymbol{a}$ (*Taylorův polynom*).
2) Pro $k = 0$ dostáváme Lagrangeovu větu:
$f(\boldsymbol{a}+\boldsymbol{h}) = f(\boldsymbol{a}) + \boldsymbol{h} \cdot \operatorname{grad} f(\boldsymbol{a}+\alpha\boldsymbol{h}) = f'_{\boldsymbol{h}}(\boldsymbol{a}+\alpha\boldsymbol{h})$.

Příklad. Pomocí Taylorova polynomu odhadněte $1{,}05^{3{,}02}$.
$f(x,y) = x^y$, $\boldsymbol{a} = (1,3)$, $\boldsymbol{h} = (0{,}05; 0{,}02)$,
$(\boldsymbol{h} \cdot \operatorname{grad}) f = h_1 \frac{\partial f}{\partial x} + h_2 \frac{\partial f}{\partial y} = h_1 y x^{y-1} + h_2 x^y \ln x = 3h_1$,
$(\boldsymbol{h} \cdot \operatorname{grad})^2 f = h_1^2 \frac{\partial^2 f}{\partial x^2} + 2h_1 h_2 \frac{\partial^2 f}{\partial x\,\partial y} + h_2^2 \frac{\partial^2 f}{\partial y^2} =$
$= h_1^2 y(y-1)x^{y-2} + 2h_1h_2 x^{y-1}(1 + y\ln x) + h_2^2 x^y \ln^2 x =$
$= 6h_1^2 + 2h_1h_2$,
$f(\boldsymbol{a}+\boldsymbol{h}) \approx f(\boldsymbol{a}) + (\boldsymbol{h} \cdot \operatorname{grad}) f(\boldsymbol{a}) + \frac{1}{2}(\boldsymbol{h} \cdot \operatorname{grad})^2 f(\boldsymbol{a}) =$
$= 1 + 3h_1 + 3h_1^2 + h_1h_2 = 1{,}1585$ ($1{,}05^{3{,}02} = 1{,}1587\ldots$)

Příklad. Určete diferenciál $f(x,y) = g(\mathrm{e}^{xy}, \mathrm{e}^{-xy})$, $g \in C^1$.
Pro $(\mathrm{e}^{xy}, e^{-xy}) = (u,v) = \big(f_1(x,y), f_2(x,y)\big) = F(x,y)$ je $f = g \circ F$

$$\mathrm{d}g(x,y) \sim \left(\tfrac{\partial g}{\partial u}, \tfrac{\partial g}{\partial v}\right)$$
$$\mathrm{d}F(x,y) \sim \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y\,\mathrm{e}^{xy} & x\,\mathrm{e}^{xy} \\ -y\,\mathrm{e}^{-xy} & -x\,\mathrm{e}^{-xy} \end{pmatrix}$$
$$\mathrm{d}f(x,y) \sim \left(y\,\mathrm{e}^{xy} \tfrac{\partial g}{\partial u} - y\,\mathrm{e}^{-xy} \tfrac{\partial g}{\partial v}, x\,\mathrm{e}^{xy} \tfrac{\partial g}{\partial u} - x\,\mathrm{e}^{-xy} \tfrac{\partial g}{\partial v}\right)$$

Věta. *Nechť $G \subset \mathbb{R}^n$, $H \subset \mathbb{R}^k$ jsou otevřené množiny, $F\colon G \to H$ má v bodě $\boldsymbol{a} \in G$ derivaci ve směru $\boldsymbol{h} \in \mathbb{R}^n$, $g\colon H \to \mathbb{R}$ má v bodě $\boldsymbol{b} = F(\boldsymbol{a})$ diferenciál. Pak funkce $(g \circ F)$ má v bodě $\boldsymbol{a}$ derivaci ve směru $\boldsymbol{h}$ a platí*

$$(g \circ F)'_{\boldsymbol{h}}(\boldsymbol{a}) = \mathrm{d}g(\boldsymbol{b})\big(F'_{\boldsymbol{h}}(\boldsymbol{a})\big)\,.$$

Důkaz (pokud F má diferenciál).

$$(g \circ F)'_{\boldsymbol{h}}(\boldsymbol{a}) = \mathrm{d}(g \circ F)(\boldsymbol{a})(\boldsymbol{h}) = \big(\mathrm{d}g(\boldsymbol{b}) \circ \mathrm{d}F(\boldsymbol{a})\big)(\boldsymbol{b}) \\ = \mathrm{d}g(\boldsymbol{b})\big(\mathrm{d}F(\boldsymbol{a})(\boldsymbol{h})\big) = \mathrm{d}g(\boldsymbol{b})\big(F'_{\boldsymbol{h}}(\boldsymbol{a})\big)$$

Derivace složené funkce

$n = 1$: $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$, $(g \circ f)'(a) = g'\big(f(a)\big) \cdot f'(a)$

Věta. *Nechť $G \subset \mathbb{R}^n$, $H \subset \mathbb{R}^k$ jsou otevřené množiny, $F\colon G \to H$ má diferenciál v bodě $\boldsymbol{a} \in G$, $g\colon H \to \mathbb{R}$ má diferenciál v bodě $\boldsymbol{b} = F(\boldsymbol{a})$. Pak funkce $(g \circ F)$ má v bodě $\boldsymbol{a}$ diferenciál a platí*

$$\mathrm{d}(g \circ F)(\boldsymbol{a}) = \mathrm{d}g(\boldsymbol{b}) \circ \mathrm{d}F(\boldsymbol{a})\,.$$

Poznámky.
1) Složení lineárních zobrazení je lineární.
2) Matice složeného zobrazení je součinem matic ($F = (f_1, \ldots, f_k)$):

$$\mathrm{d}g(\boldsymbol{b}) \sim \operatorname{grad} g(\boldsymbol{b})$$
$$\mathrm{d}F(\boldsymbol{a}) \sim \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{a}) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_k}{\partial x_n}(\boldsymbol{a}) \end{pmatrix}$$
$$\mathrm{d}(g \circ f)(\boldsymbol{a}) \sim \operatorname{grad} g(\boldsymbol{b}) \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{a}) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_k}{\partial x_n}(\boldsymbol{a}) \end{pmatrix}$$
$$= \left(\operatorname{grad} g(\boldsymbol{b})\, \tfrac{\partial f}{\partial x_1}(\boldsymbol{a}), \ldots, \operatorname{grad} g(\boldsymbol{b})\, \tfrac{\partial f}{\partial x_n}(\boldsymbol{a})\right).$$

3) Lze i pro vektorovou funkci g.
4) Pro $n = k = 1$ dostaneme násobení čísel (derivací).

Pro parciální derivace dostáváme tzv. *řetězové pravidlo* (značíme $F(\boldsymbol{x}) = \boldsymbol{y}$, $g(y_1, \ldots, y_k)$):

$$\tfrac{\partial (g\circ F)}{\partial x_i}(\boldsymbol{a}) = \operatorname{grad} g(\boldsymbol{b}) \cdot \tfrac{\partial F}{\partial x_i} \\ = \left(\tfrac{\partial g}{\partial y_1}(\boldsymbol{b}), \ldots, \tfrac{\partial g}{\partial y_k}(\boldsymbol{b})\right) \cdot \left(\tfrac{\partial f_1}{\partial x_i}(\boldsymbol{a}), \ldots, \tfrac{\partial f_k}{\partial x_i}(\boldsymbol{a})\right) \\ = \tfrac{\partial g}{\partial y_1}(\boldsymbol{b})\, \tfrac{\partial f_1}{\partial x_i}(\boldsymbol{a}) + \cdots + \tfrac{\partial g}{\partial y_k}(\boldsymbol{b})\, \tfrac{\partial f_k}{\partial x_i}(\boldsymbol{a})\,.$$

Někdy vypuštíme argumenty (uvažujeme $\boldsymbol{x}$, $\boldsymbol{y}$)

$$\frac{\partial(g \circ F)}{\partial x_i} = \frac{\partial g}{\partial y_1} \cdot \frac{\partial f_1}{\partial x_i} + \cdots + \frac{\partial g}{\partial y_k} \cdot \frac{\partial f_k}{\partial x_i}\,.$$

Někdy se nerozlišuje f_i od y_i ($f = g \circ F$):

$$\frac{\partial f}{\partial x_i} = \frac{\partial g}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_i} + \cdots + \frac{\partial g}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_i}\,.$$

(Někdy se nerozlišuje ani g od f.)

Příklad. Určete parciální derivace $f(x,y) = g(\mathrm{e}^{xy}, e^{-xy})$.
Označme $(\mathrm{e}^{xy}, e^{-xy}) = (u,v)$:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} = y\,\mathrm{e}^{xy} \frac{\partial g}{\partial u} - y\,\mathrm{e}^{-xy} \frac{\partial g}{\partial v}\,,$$
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} = x\,\mathrm{e}^{xy} \frac{\partial g}{\partial u} - x\,\mathrm{e}^{-xy} \frac{\partial g}{\partial v}\,.$$

Příklad. Pro $f(x,y,z) = x\,g(\frac{y}{x}, \frac{z}{x})$, $g \in C^1$ je
$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} + z\,\frac{\partial f}{\partial z} = f$.